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Positivity and Stability of Linear Volterra Integro-differential Equations in a Banach Lattice

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1. Introduction

Let $(X, \|\cdot\|) =: X$ be a complex Banach lattice with the real part $X_{\mathbb{R}}$ and the positive convex cone X_+ (cf. [5, Chapter C]. [8]), and $\mathcal{L}(X)$ be the space of all bounded linear operators on X. We consider an abstract Volterra integro-differential equation

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-s)x(s)ds \tag{1}$$

on X, where A is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ and $B(\cdot): \mathbb{R}_+ := [0, \infty) \to \mathcal{L}(X)$ is continuous in t with respect to the operator norm and

$$(T(t))_{t\geq 0} \text{ is a compact semigroup and } \int_0^{+\infty} \|B(t)\|dt < +\infty. \tag{2}$$

In [3], Hino and Murakami characterized the uniform asymptotic stability of the zero solution of Eq. (1) in connection with the the invertibility of the characteristic operator

$$zI - A - \int_0^{+\infty} B(t)e^{-zt}dt \qquad (I; \text{the identity operator on } X)$$

of Eq. (1) for z belonging to the closed right half plane, as well as the integrability of the resolvent for Eq. (1). In case that the space X is finite dimensional, Pham H.A. Ngoc et al. [6] studied the positivity of Eq. (1) and proved that the invertibility of the characteristic operator reduces to that of the operator $zI - A - \int_0^{+\infty} B(t)dt$, where $A + \int_0^{+\infty} B(t)dt$ is a Metzler matrix and consequently the uniform asymptotic stability of the zero solution for positive equations is equivalent to the condition which is much easier than the one for the characteristic operator in checking.

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In this paper, we will proceed with the investigation for the case that Eq. (1) is considered on a Banach lattice X, and extend several results obtained in [6] to positive systems in infinite dimensional spaces.

To make the presentation self-contained, we give some basic facts on Banach lattices which will be used in the sequel (see, e.g. [8]). Let $X_{\mathbb{R}} \neq \{0\}$ be a *real* vector space endowed with an order relation $\leq$. Then $X_{\mathbb{R}}$ is called an *ordered vector space.* Denote the *positive* elements of $X_{\mathbb{R}}$ by $X_+ := \{x \in X_{\mathbb{R}} \ : \ 0 \leq x\}$. If furthermore the *lattice property* holds, that is, if $x \vee y := \sup\{x, y\} \in X_{\mathbb{R}}$, for $x, y \in X_{\mathbb{R}}$, then $X_{\mathbb{R}}$ is called a *vector lattice.* It is important to note that X_+ is *generating,* that is,

$$X_{\mathbb{R}} = X_+ - X_+.$$

Then, the modulus of $x \in X_{\mathbb{R}}$ is defined by $|x| := x \vee (-x)$. If $\|\cdot\|$ is a norm on the vector lattice $X_{\mathbb{R}}$ satisfying the *lattice norm property,* that is, if

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|, \qquad x, y \in X_{\mathbb{R}}, \tag{3}$$

then $X_{\mathbb{R}}$ is called a *normed vector lattice.* If, in addition, $(X_{\mathbb{R}}, \|\cdot\|)$ is a Banach space then $X_{\mathbb{R}}$ is called a (real) Banach lattice.

We now extend the notion of Banach lattices to the complex case. For this extension all underlying vector lattices $X_{\mathbb{R}}$ are assumed to be *relatively uniformly complete,* that is, if for every sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfying $\sum_{n=1}^{\infty} |\lambda_n| < +\infty$ and for every $x \in X_{\mathbb{R}}$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in $X_{\mathbb{R}}$ it holds that

$$0 \leq x_n \leq \lambda_n x \Rightarrow \sup_{n\in\mathbb{N}} \Big(\sum_{i=1}^{n} x_i\Big) \in X_{\mathbb{R}}.$$

Now let $X_{\mathbb{R}}$ be a relatively uniformly complete vector lattice. The *complexification* of $X_{\mathbb{R}}$ is defined by $X = X_{\mathbb{R}} + iX_{\mathbb{R}}$. The modulus of $z = x + iy \in X$ is defined by

$$|z| = \sup_{0\leq\phi\leq 2\pi} |(\cos\phi)x + (\sin\phi)y| \in X_{\mathbb{R}}. \tag{4}$$

A complex vector lattice is defined as the complexification of a relatively uniformly complete vector lattice equipped with the modulus (4). If $X_{\mathbb{R}}$ is normed then

$$\|x\| := \||x|\|, \qquad x \in X \tag{5}$$

defines a norm on X satisfying the lattice norm property; in fact, the norm restricted to $X_{\mathbb{R}}$ is equivalent to the original norm in $X_{\mathbb{R}}$, and we use the same symbol $\|\cdot\|$ to denote the (new) norm. If $X_{\mathbb{R}}$ is a Banach lattice, then X equipped with the modulus (4) and the norm (5) is called a complex Banach lattice.

Throughout this paper, X is assumed to be a complex Banach lattice with the real part $X_{\mathbb{R}}$ and the positive convex cone X_+. Let $T \in \mathcal{L}(X)$. Then T is called *real* if $T(X_{\mathbb{R}}) \subset X_{\mathbb{R}}$.

A real operator T is called *positive* and denoted by $T \geq 0$ if $T(X_+) \subset X_+$. By $S \leq T$ we mean $T - S \geq 0$, for $T, S \in \mathcal{L}(X)$. We introduce the notation

$$\mathcal{L}_+(X) := \{T \in \mathcal{L}(X) \ : \ T \geq 0\}. \tag{6}$$

For $T \in \mathcal{L}_+(X)$, we emphasize the simple but important fact

$$\|T\| = \sup_{x \in X_+, \|x\|=1} \|Tx\|, \tag{7}$$

see e.g. [8, p.230]. A C_0 semigroup $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called *positive* if $T(t) \in \mathcal{L}_+(X)$ for all $t \geq 0$.

2. Characterizations of positive linear Volterra integro-differential equations in Banach lattices

In this section, we will introduce the notion of positivity for Eq. (1), and give a characterization of positivity of Eq. (1) in terms of positivity of the semigroup $(T(t))_{t\geq 0}$ and of the kernel function $B(\cdot)$.

For any $(\sigma, \phi) \in \mathbb{R}_+ \times C([0,\sigma], X)$, there exists a unique continuous function $x : \mathbb{R}_+ \to X$ such that $x \equiv \phi$ on $[0, \sigma]$ and the following relation holds:

$$x(t) = T(t-\sigma)\phi(\sigma) + \int_\sigma^t T(t-s)\Big\{\int_0^s B(s-\tau)x(\tau)d\tau\Big\}ds, \qquad t \geq \sigma,$$

see e.g. [2]. The function x is called a (*mild*) solution of Eq. (1) through (σ, ϕ) on $[\sigma, \infty)$, and denoted by $x(\cdot, \sigma, \phi)$.

We say that Eq. (1) is *positive* if $x(t, \sigma, \phi) \in X_+$ on $[\sigma, \infty)$ whenever $(\sigma, \phi) \in \mathbb{R}_+ \times C([0, \sigma], X_+)$.

Theorem 1. *If A generates a positive semigroup $(T(t))_{t\geq 0}$ on X and $B(t) \geq 0$ for any $t \geq 0$ then Eq. (1) is positive. Conversely, if Eq. (1) is positive and A is the infinitesimal generator of a positive C_0 semigroup $(T(t))_{t\geq 0}$ on X then $B(t) \geq 0$ for each $t \geq 0$.*

Proof. The former part of the theorem can be proved by the standard argument; so we will omit the proof. In the following, we will prove the latter part of the proof. To do this, we will firstly check that $B(t)$ is real for each $t \geq 0$. Let any $\sigma > 0$ and $a \in X_+$ be given. For each integer n such that $1/n < \sigma$, we consider a function $\phi_n \in C([0, \sigma], X_+)$ defined by $\phi_n(t) = a$ if $t \in [0, \sigma - 1/n]$, and $\phi_n(t) = n(\sigma - t)a$ if $t \in (\sigma - 1/n, \sigma]$. By the positivity of Eq. (1), we get $x(t, \sigma, \phi_n) \geq 0$ for any $t \geq \sigma$, and hence

$$\begin{aligned}(1/h)x(h+\sigma, \sigma, \phi_n) &= \frac{1}{h}\Big(T(h)\phi_n(\sigma) + \int_\sigma^{\sigma+h} T(h+\sigma-s)\big(\int_0^s B(s-\tau)x(\tau, \sigma, \phi_n)d\tau\big)ds\Big)\\ &= \frac{1}{h}\int_\sigma^{\sigma+h} T(h+\sigma-s)\big(\int_0^s B(s-\tau)x(\tau, \sigma, \phi_n)d\tau\big)ds\\ &\geq 0\end{aligned}$$

for any $h > 0$. Observe that

$$\lim_{h \to +0} \left[\frac{1}{h} \int_{\sigma}^{\sigma+h} T(h+\sigma-s) (\int_0^s B(s-\tau) x(\tau, \sigma, \phi_n) d\tau) ds \right]$$
$$= \int_0^{\sigma} B(\sigma-\tau) x(\tau, \sigma, \phi_n) d\tau = \int_0^{\sigma} B(\sigma-\tau) \phi_n(\tau) d\tau.$$

Hence it follows that

$$\int_0^{\sigma} B(\sigma-\tau)\phi_n(\tau) d\tau \geq 0.$$

Letting $n \to \infty$ in the above, we get $\int_0^{\sigma} B(\sigma-\tau) a d\tau \geq 0$ or $\int_0^{\sigma} B(s) a ds \geq 0$. Then

$$\int_t^{t+h} B(s) a ds = \int_0^{t+h} B(s) a ds - \int_0^t B(s) a ds \in X_+ - X_+ = X_{\mathbb{R}}$$

for any $t \geq 0$ and $h > 0$; consequently,

$$B(t)a = \lim_{h \to +0} \left(\frac{1}{h} \int_t^{t+h} B(s) a ds \right) \in X_{\mathbb{R}}, \quad a \in X_+.$$

Therefore it follows that $B(t) X_{\mathbb{R}} \subset X_{\mathbb{R}}$, which means that $B(t)$ is real for each $t \geq 0$.

Secondly, we will establish that $B(t) \geq 0$ for each $t \geq 0$. Let $(\sigma, \phi) \in \mathbb{R}_+ \times C([0, \sigma], X_+)$ with $\phi(\sigma) = 0$ be given. By the positivity of Eq. (1), we have $y(t) := x(t+\sigma, \sigma, \phi) \geq 0$ on $[0, \infty)$. Observe that y satisfies the relation

$$y(t) = T(t)\phi(\sigma) + \int_{\sigma}^{t+\sigma} T(t+\sigma-s) \Big\{ \int_0^s B(s-\tau) x(\tau) d\tau \Big\} ds$$
$$= \int_0^t T(t-u) \Big\{ \int_0^{\sigma+u} B(\sigma+u-\tau) x(\tau) d\tau \Big\} du = \int_0^t T(t-u) p(u) du,$$

for $t \geq 0$, where

$$p(u) := \int_0^{\sigma+u} B(\sigma+u-\tau) x(\tau) d\tau.$$

Now, let us take a real number λ sufficiently large such that $\sup_{t \geq 0} \left(e^{(-\lambda+1)t} \|T(t)\| \right) < \infty$. Then $\lambda \in \rho(A)$ (the resolvent set of A), and $R(\lambda, A) := (\lambda I - A)^{-1}$ is given by

$$R(\lambda, A) x = \int_0^{\infty} e^{-\lambda t} T(t) x dt, \quad x \in X.$$

Therefore it follows that $\lambda \in \rho(A^*)$ and $R(\lambda, A^*) = R(\lambda, A)^*$. Let v_+^* be an arbitrary element in $(X^*)_+$, the space of all positive bounded linear functionals on X, and set $v^* = R(\lambda, A^*) v_+^*$. Then $v^* \in \mathcal{D}(A^*)$ and

$$\langle v^*, y(t) \rangle = \langle v^*, \int_0^t T(t-u) p(u) du \rangle, \quad t \geq 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical duality pairing of X^* and X. Since $y(t) \geq 0$, the positivity of $(T(t))_{t \geq 0}$ implies that

$$R(\lambda, A) y(t) = \int_0^{\infty} e^{-\lambda u} T(u) y(t) du \geq 0,$$

and hence $\langle v^*, y(t)\rangle = \langle v_+^*, R(\lambda, A)y(t)\rangle \geq 0$ by the fact that $v_+^* \geq 0$. Consequently, $(d^+/dt)\langle v^*, y(t)\rangle|_{t=0} \geq 0$ by the fact that $\langle v^*, y(0)\rangle = v^*(0) = 0$. Notice that $AR(\lambda, A) = -I + \lambda R(\lambda, A)$. Therefore it follows that

$$(AR(\lambda, A))^* = -I^* + \lambda R(\lambda, A)^* = -I^* + \lambda R(\lambda, A^*) = A^* R(\lambda, A^*),$$

and hence

$$\begin{aligned}
&\frac{d^+}{dt}\langle v^*, \int_0^t T(t-u)p(u)du\rangle = \frac{d^+}{dt}\langle v_+^*, R(\lambda, A)\int_0^t T(t-u)p(u)du\rangle \\
&= \lim_{h\to+0}(1/h)\left\{\langle v_+^*, R(\lambda, A)\int_0^{t+h} T(t+h-u)p(u)du - R(\lambda, A)\int_0^t T(t-u)p(u)du\rangle\right\} \\
&= \lim_{h\to+0}\left\{\langle v^*, (1/h)\int_t^{t+h} T(t+h-u)p(u)du\rangle\right. \\
&\qquad \left. + \langle v_+^*, R(\lambda, A)\frac{T(h)-I}{h}\int_0^t T(t-u)p(u)du\rangle\right\} \\
&= \langle v^*, p(t)\rangle + \langle v_+^*, AR(\lambda, A)\int_0^t T(t-u)p(u)du\rangle \\
&= \langle v^*, p(t)\rangle + \langle (AR(\lambda, A))^* v_+^*, y(t)\rangle \\
&= \langle v^*, p(t)\rangle + \langle A^* R(\lambda, A^*)v_+^*, y(t)\rangle \\
&= \langle v^*, p(t)\rangle + \langle A^* v^*, y(t)\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d^+}{dt}\langle v^*, y(t)\rangle|_{t=0} &= \langle v^*, p(0)\rangle + \langle A^* v^*, y(0)\rangle = \langle v^*, \int_0^\sigma B(\sigma-\tau)x(\tau)d\tau\rangle \\
&= \langle R(\lambda, A)^* v_+^*, \int_0^\sigma B(\sigma-\tau)\phi(\tau)d\tau\rangle \\
&= \langle v_+^*, R(\lambda, A)\int_0^\sigma B(\sigma-\tau)\phi(\tau)d\tau\rangle,
\end{aligned}$$

and consequently

$$\langle v_+^*, R(\lambda, A)\int_0^\sigma B(\sigma-\tau)\phi(\tau)d\tau\rangle \geq 0.$$

Rewriting $\phi(s-\tau)$ as $\psi(\tau)$, we obtain

$$\langle v_+^*, R(\lambda, A)\int_0^\sigma B(u)\psi(u)du\rangle \geq 0 \tag{8}$$

for any $v_+^* \in (X^*)_+$ and any $\psi \in C([0,\sigma]; X_+)$ with $\psi(0) = 0$. We claim that

$$R(\lambda, A)B(t)a \geq 0 \qquad (\forall\, t \in (0,\sigma],\ a \in X_+). \tag{9}$$

Assume that the claim is false. Then there are $t_1 \in (0,\sigma]$ and $a \in X_+$ such that $R(\lambda, A)B(t_1)a \notin X_+$. Notice that $R(\lambda, A)B(t_1)a \in X_{\mathbf{R}}$ by $R(\lambda, A) \geq 0$ and $B(t)a \in X_{\mathbf{R}}$. Since X_+ is a closed convex cone, the well known result in functional analysis (e.g., [4, Chapter 3, Theorem 6]) yields that there exists a $v_+^* \in X^*$ with the property that $v_+^* \geq 0$ on X_+ and $\langle v_+^*, R(\lambda, A)B(t_1)a\rangle < 0$. Hence $v_+^* \in (X^*)_+$, and moreover there exists an interval $[c,d] \subset (0,\sigma)$ satisfying $\langle v_+^*, R(\lambda,A)B(t)a\rangle < 0$ for all $t \in [c,d]$. Then one can choose a nonnegative scalar continuous function χ so that $\chi(0) = 0$ and

$$\langle v_+^*, \int_0^\sigma R(\lambda,A)B(t)\chi(t)adt\rangle = \int_0^\sigma \langle v_+^*, R(\lambda,A)B(t)a\rangle\chi(t)dt < 0;$$

which leads to a contradiction by considering $\chi(t)a$ as $\psi(t)$ in (8).

Finally, $B(t) \geq 0$ immediately follows from (9) and the fact that $\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x$ for any $x \in X$. The proof is completed.

3. Stability of positive linear Volterra integro-differential equations in Banach lattices

In this section, we continue to assume that (2) is valid, and investigate the uniform asymptotic stability property of the zero solution of Eq. (1). Before stating the main result of this section, we introduce some notations. For the C_0-semigroup $(T(t))_{t\geq 0}$ with the infinitesimal generator A, we consider the following quantities:

(i) The *spectral bound*,

$$s(A) := \sup\{\Re\lambda \;\; \lambda \in \sigma(A)\},$$

where $\sigma(A)$ is spectrum of the linear operator A.

(ii) The *growth bound* $\omega(A)$,

$$\omega(A) := \inf\{\omega \in \mathbb{R} \;:\; \text{there exists } M > 0 \text{ such that}$$

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0\}.$$

It is well-known that

$$-\infty \leq s(A) \leq \omega(A) < +\infty, \tag{10}$$

see, e.g [1], [5].

In what follows, we will essentially use the following two results.

Theorem 2. [3] *The following statements are equivalent:*

(i) The zero solution of Eq. (1) is uniformly asymptotically stable.

(ii) The operator $\lambda I - A - \int_0^{+\infty} e^{-\lambda s}B(s)ds$ is invertible in $\mathcal{L}(X)$ for any $\lambda \in \mathbb{C}, \Re\lambda \geq 0$.

Lemma 1. *Assume that A generates a positive semigroup $(T(t))_{t\geq 0}$ on X and $P \in \mathcal{L}(X), Q \in \mathcal{L}_+(X)$. If*

$$|Px| \leq Q|x|, \; \forall x \in X,$$

then

$$\omega(A+P) = s(A+P) \leq s(A+Q) = \omega(A+Q).$$

Proof of Lemma 1. Let $(G(t))_{t\geq 0}$ and $(H(t))_{t\geq 0}$ be the C_0 semigroups with the infinitesimal generators $A+P$ and $A+Q$, respectively. Since A generates the compact semigroup $(T(t))_{t\geq 0}$, so do $A+P$ and $A+Q$, see e.g. [1, 5]. This implies that $s(A+P)=\omega(A+P)$ and $s(A+Q) = \omega(A+Q)$, see e.g. [1, 5]. As the standard property of C_0 compact semigroups, we know that $e^{\sigma(C)} = \sigma\{M(1)\}\backslash\{0\}$, where C is the infinitesimal generator of any compact C_0 semigroup $(M(t))_{t\geq 0}$ on X; see e.g. [1, Corollary IV.3.11]. Hence we have $e^{\omega(C)} = r(M(1))$, where $r(M(1))$ is the spectral radius of the operator $M(1)$. Thus, it is sufficient to show that

$$r(G(1)) \leq r(H(1)).$$

Note that $(G(t))_{t\geq 0}$ and $(H(t))_{t\geq 0}$ are defined respectively by

$$G(t)x = \lim_{n\to+\infty}(T(t/n)e^{(t/n)P})^n x, \quad H(t)x = \lim_{n\to+\infty}(T(t/n)e^{(t/n)Q})^n x, \quad x \in X,$$

for each $t \geq 0$; see e.g. [5, p.44] and see also [1, Theorem III.5.2]. By the positivity of $(T(t))_{t\geq 0}$ and the hypothesis of $|Px| \leq Q|x|$, $x \in X$, it is easy to see that

$$|G(1)x| \leq H(1)|x|, \quad x \in X.$$

Then, we get further that

$$|G(1)^k x| \leq H(1)^k |x|, \quad x \in X, k \in \mathbb{N}, \tag{11}$$

by induction. From the property of a norm on Banach lattices (3), it follows from (11) and (7) that

$$\|G(1)^k\| \leq \|H(1)^k\|.$$

By the well-known Gelfand 's formula, we have

$$r(G(1)) \leq r(H(1)),$$

which completes our proof.

We are now in the position to prove the main result of this section.

Theorem 3. *Assume that A generates a positive semigroup $(T(t))_{t\geq 0}$ on X and $B(t) \geq 0$ for all $t \geq 0$. Then the following two statements are equivalent:*

(i) The zero solution of Eq. (1) is uniformly asymptotically stable.

(ii) $s(A + \int_0^{+\infty} B(\tau)d\tau) < 0$.

Proof. (ii) $\Rightarrow$ (i) Assume that $\lambda I - A - \int_0^{+\infty} e^{-\lambda s}B(s)ds$ is not invertible for some $\lambda \in \mathbb{C}$, $\Re\lambda \geq 0$. This implies that $\lambda \in \sigma(A + \int_0^{+\infty} e^{-\lambda s}B(s)ds)$. We thus get

$$0 \leq \Re\lambda \leq s(A + \int_0^{+\infty} e^{-\lambda s}B(s)ds).$$

On the other hand, it is easy to see that

$$\left|\left(\int_0^{+\infty} e^{-\lambda s}B(s)ds\right)x\right| \le \int_0^{+\infty} B(s)ds|x|,$$

by the hypothesis of $B(t) \ge 0, \forall t \ge 0$. Hence, we get

$$0 \le s(A + \int_0^{+\infty} e^{-\lambda s}B(s)ds) \le s(A + \int_0^{+\infty} B(s)ds),$$

by Lemma 1. This is a contradiction to the assumption that $s(A + \int_0^{+\infty} B(s)ds) < 0$.

(i) $\Longrightarrow$ (ii) For every $\lambda \ge 0$, we put $\Phi_\lambda = \int_0^\infty B(t)e^{-\lambda t}dt$ and $f(\lambda) = s(A+\Phi_\lambda)$. Consider the real function defined by $g(\lambda) := \lambda - f(\lambda), \lambda \ge 0$. We show that $g(0) = -s(A+\Phi_0) > 0$. Since $B(\cdot)$ is positive, by almost the same argument as in [1, Proposition VI.6.13] one can see that $f(\lambda)$ is non-increasing and left continuous in $\lambda > 0$. Hence $g(\lambda)$ is increasing and left continuous in λ with $\lim_{\lambda\to+\infty} g(\lambda) = +\infty$. We assert that the function $g(\lambda)$ is right continuous in $\lambda \ge 0$. Indeed, if this assertion is false, then there is a $\lambda_0 \ge 0$ such that $(s^+ :=) \lim_{\varepsilon\to+0} f(\lambda_0 + \varepsilon) < f(\lambda_0) =: s_0$. Notice that $s_0 = s(A + \Phi_{\lambda_0})$ and $A + \Phi_{\lambda_0} =: \tilde{A}$ generates a positive and compact C_0 semigroup $(e^{\tilde{A}t})_{t\ge 0}$. It follows that $s_0 = s(\tilde{A}) \in \sigma(\tilde{A})$ by [1, Theorem VI.1.10]. Take a $t_0 \in \rho(\tilde{A})$. Since

$$\sigma(R(t_0, \tilde{A}))\backslash\{0\} = \{\frac{1}{t_0 - \mu} \mid \mu \in \sigma(\tilde{A})\}$$

by [1, Theorem IV.1.13], we get $1/(t_0 - s_0) \in \sigma(R(t_0, \tilde{A}))$. Observe that $1/(t_0 - s_0)$ is isolated in the spectrum $\sigma(R(t_0, \tilde{A}))$ of the compact operator $R(t_0, \tilde{A})$. Therefore, if s_1 is sufficiently close to s_0 and $s_1 \ne s_0$, then $1/(t_0 - s_1)$ is sufficiently close to $1/(t_0 - s_0)$; hence $1/(t_0 - s_1) \notin \sigma(R(t_0, \tilde{A}))$, in particular, $s_1 \notin \sigma(\tilde{A})$. Therefore one can choose an $s_1 \in (s^+, s_0)$ so that $s_1 \in \rho(\tilde{A})$, that is, $s_1 I - A - \Phi_{\lambda_0}$ has a bounded inverse $(s_1 I - A - \Phi_{\lambda_0})^{-1}$ in $\mathcal{L}(X)$. In the following, we will show that $(s_1 I - A - \Phi_{\lambda_0})^{-1} \ge 0$. Since $s^+ < s_1$, it follows that $s(A + \Phi_{\lambda_0+\varepsilon}) < s_1$ for small $\varepsilon > 0$. Then [1, Lemma VI.1.9] implies that $(s_1 I - A - \Phi_{\lambda_0+\varepsilon})^{-1} \ge 0$ and

$$(s_1 I - A - \Phi_{\lambda_0+\varepsilon})^{-1}x = \int_0^\infty e^{-s_1 t}\exp((A + \Phi_{\lambda_0+\varepsilon})t)x dt, \quad x \in X.$$

Observe that

$$\begin{aligned} s_1 I - A - \Phi_{\lambda_0+\varepsilon} &= s_1 I - A - \Phi_{\lambda_0} + (\Phi_{\lambda_0} - \Phi_{\lambda_0+\varepsilon}) \\ &= \Big(I - (\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1, \tilde{A})\Big)(s_1 I - \tilde{A}) \end{aligned}$$

and that

$$\begin{aligned} &\|(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1, \tilde{A})\| \\ &\le \int_0^\infty \|B(\tau)e^{-\lambda_0\tau}(1 - e^{-\varepsilon\tau})\|d\tau\|R(s_1, \tilde{A})\| \\ &\le \int_0^\infty \|B(\tau)\|(1 - e^{-\varepsilon\tau})d\tau\|R(s_1, \tilde{A})\| \to 0 \quad (\varepsilon \to +0). \end{aligned}$$

Hence, if $\varepsilon > 0$ is small, then $\|(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1, \tilde{A})\| < 1/2$; hence $I - (\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1, \tilde{A})$ is invertible with

$$\left(I - (\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1, \tilde{A})\right)^{-1} = \sum_{n=0}^{\infty} \left\{(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1, \tilde{A})\right\}^n,$$

and consequently

$$(s_1 I - A - \Phi_{\lambda_0+\varepsilon})^{-1} = R(s_1, \tilde{A}) \sum_{n=0}^{\infty} \left\{(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1, \tilde{A})\right\}^n.$$

Thus we get

$$\begin{aligned}
&\|(s_1 I - A - \Phi_{\lambda_0+\varepsilon})^{-1} - (s_1 I - A - \Phi_{\lambda_0})^{-1}\| \\
&= \|R(s_1, \tilde{A}) \sum_{n=1}^{\infty} \left\{(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1, \tilde{A})\right\}^n \| \\
&\le \|R(s_1, \tilde{A})\| \sum_{n=1}^{\infty} \|(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1, \tilde{A})\|^n \\
&= \|R(s_1, \tilde{A})\| \|(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1, \tilde{A})\| / (1 - \|(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1, \tilde{A})\|) \\
&\le 2\|R(s_1, \tilde{A})\|^2 \int_0^{\infty} \|B(\tau)\|(1 - e^{-\varepsilon\tau})d\tau \to 0 \quad (\varepsilon \to +0).
\end{aligned}$$

Then the positivity of $(s_1 I - A - \Phi_{\lambda_0})^{-1}$ follows from the positivity of $(s_1 I - A - \Phi_{\lambda_0+\varepsilon})^{-1}$, as desired. Applying [1, Lemma VI.1.9] again, we get $s_1 > s(A+\Phi_{\lambda_0}) = s_0$, a contradiction to the fact that $s_1 < s_0$. Thus, $f(\lambda)$ and $g(\lambda)$ must be right continuous in $\lambda \ge 0$.

Assume contrary that $g(0) \le 0$. Since the function g is continuous on $[0, \infty)$ and $\lim_{\lambda\to\infty} g(\lambda) = \infty$, there is a $\lambda_1 \ge 0$ such that $g(\lambda_1) = 0$; that is, $\lambda_1 = s(A + \Phi_{\lambda_1})$.

Since $A + \Phi_{\lambda_1}$ generates a positive semigroup and $s(A + \Phi_{\lambda_1}) > -\infty$, by virtue of [1, Theorem VI.1.10] $\lambda_1 = s(A + \Phi_{\lambda_1}) \in \sigma(A + \Phi_{\lambda_1})$. Since $A + \Phi_{\lambda_1}$ generates a compact C_0 semigroup, it follows from [1, Corollary IV.1.19] that $\sigma(A + \Phi_{\lambda_1})$ is identical with $P_\sigma(A + \Phi_{\lambda_1})$, the point spectrum of $A + \Phi_{\lambda_1}$. Thus, there exists a nonzero $x_1 \in X$ such that $(A + \Phi_{\lambda_1})x_1 = \lambda_1 x_1$; that is, $Ax_1 + \int_0^{+\infty} B(\tau)e^{-\lambda_1\tau}x_1 d\tau = \lambda_1 x_1$. Put $x(t) = e^{\lambda_1 t}x_1$ for $t \in \mathbb{R}$. Then, it is easy to see that

$$\dot{x}(t) = Ax(t) + \int_0^{+\infty} B(\tau)x(t-\tau)d\tau, \quad t \in \mathbb{R};$$

hence x satisfies the "limiting" equation of Eq. (1). By virtue of [3, Proposition 2.3], the zero solution of the limiting equation is uniformly asymptotically stable because of the uniform asymptotic stability of Eq. (1). Hence we must get $\lim_{t\to\infty} \|x(t)\| = 0$. However, $\|x(t)\| = e^{\lambda_1 t}\|x_1\| \ge \|x_1\| > 0$ for $t \ge 0$, a contradiction. This completes the proof of the implication (i) $\Longrightarrow$ (ii).

References

1. K.J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, G.T.M., vol. 194, Springer-Verlag, 2000.

2. H.R. Henriquez, Periodic solutions of quasi-linear partial functional differential equations with unbounded delay, *Funkcial. Ekvac.* **37** (1994), 329–343.
3. Y. Hino and S. Murakami, Stability properties of linear Volterra integrodifferential equations in a Banach space, Funkcial. Ekvac., **48** (2005), 367–392.
4. L.V. Kantorovich and G.P. Akilov, *Functional Analysis*, Pergamon Press, 1982.
5. R. Nagel (ed.), *One-parameter Semigroups of Positive Operators*, Lect. Notes in Math., vol. 1184, Springer-Verlag, 1986.
6. Pham Huu Anh Ngoc, Toshiki Naito, Jong Son Shin and Satoru Murakami, On stability and robust stability of positive linear Volterra differential equations, SIAM J. Control Optim., (in press).
7. W. Rudin, *Functional Analysis*, McGraw-Hill, New Delhi, 1988.
8. H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer-VerLag, 1974.